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Physics Letters B 569 (2003) 85–89
PHYSICS LETTERS B
www.elsevier.com/locate/npe

PCT theorem in field theory on non-commutative space

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Received 19 May 2003; received in revised form 30 June 2003; accepted 1 July 2003

Editor: M. Cvetič

Abstract

The PCT theorem is shown to be valid in quantum field theory formulated on non-commutative space–time by exploiting the properties of the Wightman functions defined in such a set up.

Quantum field theory provides the general basis and correct theoretical framework known till date to describe particle properties and interactions and the PCT theorem enjoys a unique status in any viable and physically appealing quantum theory. Stated crudely, it demands the invariance of the theory under the joint action (in any order) of the three discrete symmetries—parity (P), charge conjugation (C) and time reversal (T), and predicts the existence of an anti-unitary operator realizing this joint symmetry. In an attempt to obtain the spin-statistics connection, similar to that of Pauli [1], Schwinger [2] assumed some form of PCT invariance. Without considering parity violation, it was noted by Luders [3] that charge conjugation and time reversal impose the same restrictions on the theory, i.e., if a relativistic quantum field theory has space inversion then it must have the product of charge conjugation and time reversal as a symmetry. However, it was realized by Pauli [4] that PCT itself is always a symmetry. There are two approaches to prove or check the validity of this theorem:

(A1) It can be shown that the product of P, C, T taken in any order is same as Strong Reflection (SR) followed by Hermitian conjugation. By this we imply the following set of conditions:

$$\begin{aligned} &x \to -x, \\ &O_{\mu_1\mu_2\cdots\mu_n}(x) \to (-1)^n O_{\mu_1\mu_2\cdots\mu_n}(-x), \\ &\psi(x) \to i\gamma_5\psi(-x), \end{aligned} \tag{1}$$

where $O_{\mu_1\mu_2\cdots\mu_n}(x)$ denotes a bosonic field carrying n number of tensorial indices. Given these transformation rules, and given the fact that PCT is same as SR followed by Hermitian conjugation, it becomes very easy to check whether a given theory (with interactions) preserves PCT symmetry.

(A2) The second approach is to proceed via the axiomatic field theory route [5]. In this case, the whole theory can be reconstructed in terms of the Wightman functions defined as the vacuum expectation values of products of fields (generically denoted as $O_i(x_i)$) without worrying about

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doi:10.1016/j.physletb.2003.07.004

the Lorentz indices)

$$W(x_1, x_2, \ldots, x_n) = \langle 0|O_1(x_1)O_2(x_2)\cdots O_n(x_n)|0\rangle \qquad (2)$$

satisfying very general conditions.

The basis of formulation of a quantum field theory, whether it is the Lagrange formalism or the axiomatic formalism, is a set of axioms. Without going into details of the proofs and generalities of the axioms, we quickly review the important features and summarize them as follows:

(Ax1) The states are described by unit rays in a physical Hilbert space and the state space possesses relativistic invariance.
(Ax2) The spectral condition implying that the physical four momenta lie in or on the positive light cone.
(Ax3) Existence of a unique vacuum.
(Ax4) Notion of a quantum field and its domain.
(Ax5) Poincaré invariance of the fields, i.e., the fields transform in a fixed manner under SL(2,C).
(Ax6) Local commutativity (sometimes also known as micro-causality) implying commutation (or anti-commutation for fermions) of any two field components for space-like regions.
(Ax7) The condition of asymptotic completeness, i.e., demanding on physical grounds the validity of the following relation: $\mathcal{H} = \mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}$, where in and out refer to the incoming and outgoing collision states.

It can be shown on general grounds that a quantum theory which satisfies all these axioms respects the PCT symmetry and the normal spin-statistics relations. The reason why PCT holds a very sacred place in any quantum field theory is that apart from being the outcome of very general features in the theoretical formulation, till date no experiment has found deviations from the consequences of this result, namely equality of mass and life times for particle and its anti-particle [6]. In spite of these reasons, this area has attracted a lot of attention (for an overview see [7]). It has been realized that the requirement of local commutativity (or locality) is too strong a condition and it should suffice to prove the PCT theorem if this condition is relaxed to weak local commutativity (WLC) which, for two fields, reads

$$\langle 0|\big[O_1(x), O_2(y)\big]|0\rangle = 0, \qquad (x-y)^2 < 0 \qquad (3)$$

for any two bosonic operators (and the commutator → anti-commutator for fermions). Therefore in the modified form, the PCT theorem can be stated as the equivalence of weak local commutativity to the existence of a PCT operator satisfying the usual properties and leaving the vacuum invariant. In general WLC implies (modulo a sign factor arising due to and depending on the number of fermions permuted)

$$\langle 0|O_1(x_1)O_2(x_2)\cdots O_n(x_n)|0\rangle = \langle 0|O_n(x_n)\cdots O_2(x_2)O_1(x_1)|0\rangle \qquad (4)$$

if the set $\{x_i\}$ is a Jost point meaning thereby that $(\sum_i \lambda_i\xi_i)^2 < 0$ for real $\xi_i = x_i - x_{i+1}$ and λ_i being set of real non-negative numbers not all zero.

Our usual notions of the space–time being described by a suitable manifold and the points on it being labelled by a finite number of real coordinates may not be completely correct at smaller and smaller distances (or large enough energies), implying that the assumption of space–time being a continuum may not be valid at all scales. If that is true then the underlying theory has an intrinsic length scale involved which is usually associated with the Planck length. In fact, with the aim to circumvent the problem of infinities in quantum field theories, Snyder [8] showed that there exists a Lorentz invariant space–time with a natural unit of length. The consequences of such a solution are the modifications in the commutation relations for the operators corresponding to both coordinates and momenta. However, it is easily seen that such modifications only show up at extremely large energy/momentum scales and the low-energy physics is well described by the ordinary quantum theory. Motivated by some recent string theory arguments, the field theory formulation on the non-commutative spaces has attracted a lot of attention. For a review of field theories on non-commutative spaces and various related issues see [9]. In a non-commutative set up the usual notion of coordinates being commutative is given up and the Hermitian coordinate operators are assumed to satisfy the following commutation relation

$$\big[\hat{x}^\mu, \hat{x}^\nu\big] = i\Theta^{\mu\nu}, \qquad (5)$$

where $\Theta^{\mu\nu}$ is a real anti-symmetric matrix. In the quantum theory, this requirement translates into the fact that the product of operators be replaced by the Weyl–Moyal product (also called star product, denoted by $*$). For a constant $\Theta^{\mu\nu}$ this implies that for two operators, the star product is given by

$$f(x) * g(x) = \left[\exp\left(\frac{i}{2}\Theta^{\mu\nu}\partial_{\eta_\mu}\partial_{\zeta_\nu}\right) f(x+\eta) \times g(x+\zeta)\right]_{\eta=\zeta=0}. \tag{6}$$

From the very expression it is obvious that theories on such spaces must be highly non-local due to the presence of infinite number of derivatives. Also, such theories violate the Lorentz invariance in the sense of "particle Lorentz invariance" [7] while the observer Lorentz transformations remain the symmetry of the theory.

It is instructive to explore the validity of PCT theorem in a non-commutative theory where the locality is lost from the very beginning. However, it seems natural to assume the validity of other axioms. In particular, since we would like the theory to be describing physical situations, axioms (Ax1)–(Ax5) should still hold and this is a reasonable assumption to start with. Also, the requirement of asymptotic completeness can be demanded on physical grounds. Therefore, but for the locality condition, other axioms are satisfied and it is thus left to check whether with this condition not met, can PCT still hold. Using the first approach (A1), the authors of [10] have shown that PCT theorem is valid for any form of non-commutativity.[1] In the present note, we would like to follow the other approach (A2) and see whether the same conclusions can be reached by employing Wightman functions.

In analogy with the Wightman function defined in the ordinary theory Eq. (2), we define the Wightman function in the non-commutative theory as follows

$$W(x_1, x_2, \ldots, x_n; *) = \langle 0|O_1(x_1)O_2(x_2)\cdots O_n(x_n)|0\rangle_*, \tag{7}$$

where $*$ refers to the product of fields defined appropriately in the non-commutative theory. In our calculations we will work only with scalar fields. Generalization to fermions or vector particles is straight forward and the basic essence is the same. In the ordinary theory the commutator of two scalar fields vanishes for space-like separations (locality implies weak local commutativity and thus PCT). This is clearly not the case here. Therefore, the main task is to examine the weak local commutativity condition and see what we get out of it. Before exploring the general n point Wightman function, we take a close look at the two point Wightman function defined as

$$W_{AB}(x, y; *) \equiv \langle 0|\phi_A(x)\phi_B(y)|0\rangle_* \tag{8}$$

and the expectation value of the commutator

$$\langle 0|\big[\phi_A(x), \phi_B(y)\big]_*|0\rangle, \tag{9}$$

where again the subscript $*$ reminds of the fact that the products involved are the appropriate products referred to as above. Eq. (9) is nothing but the spectral representation. In the ordinary field theory it reads (for the same fields, i.e., for A and B to be the same) [14]

$$\langle 0|\big[\phi(x), \phi(y)\big]|0\rangle = i\int_0^\infty dm'^2\,\sigma\big(m'^2\big)\Delta(x-y; m'),$$

where $\sigma(q^2)$ is related to the spectral density $\rho(q)$ in the usual way ($\rho(q) = \sigma(q^2)\theta(q^0)$ because of Lorentz invariance and $\theta(q^0)$ is the step function) and $i\Delta(x-y; m')$ is the free field commutator which vanishes outside the light cone. However, due to presence of non-commutativity the symmetry may be more restricted and therefore in the non-commutative case we expect the non-commutative parameter to show up. On general grounds it can be argued that the commutator in the non-commutative case should have the following spectral representation

$$\langle 0|\big[\phi(x), \phi(y)\big]_*|0\rangle = i\int_0^\infty dm'^2\,\sigma\big(m'^2, i\tilde{\partial}^2\big)\Delta(\xi; m'), \tag{10}$$

[1] For non-commutative QED the discrete symmetries and their joint action has been discussed in [11]. Using the arguments based on Lorentz violation, the authors of [12] conclude that PCT is preserved in any realistic theory. The C, P and T transformation properties and hermiticity of the Seiberg–Witten maps has been discussed in [13].

where $\tilde{\partial}^\mu = \Theta^{\mu\nu}\partial_\nu$ and $\xi = x - y$. Being able to write in this form is the consequence of translational invariance which is not lost as Θ is independent of space–time variables. This is same as the result obtained in [15]. Because of the presence of derivatives in Eq. (10) it is not very clear whether the commutator expectation value will vanish outside the light cone or not. However, it does vanish outside the light cone and the reason is not hard to see. The derivatives (occurring as $\tilde{\partial}^2$) act only on $\Delta(\xi; m')$ and this being a simple function of exponentials gives back the same function with multiplicative factors proportional to $\tilde{p}^2$. For the nth term we will typically have

$$\left(\tilde{\partial}^2\right)^n \Delta(\xi; m') \propto \left(\tilde{p}^2\right)^n \Delta(\xi; m'), \tag{11}$$

thus still preserving the space–time dependence of $\Delta(\xi; m')$. The net result of all such terms can be put in a series and we have the final form

$$\langle 0|\left[\phi(x), \phi(y)\right]_*|0\rangle \sim \sum_{n=0}^{\infty} \frac{(-i\tilde{p}^2)^n}{n!} i \int_0^\infty dm'^2\, \sigma\left(m'^2\right) \Delta(\xi; m'). \tag{12}$$

Therefore, even in the non-commutative case, the right-hand side vanishes for $\xi^2 < 0$.[2] We therefore have the result that WLC holds for the field theory formulated on non-commutative space and therefore this result can be extended to non-identical fields and an arbitrary number of them. Specifically for the two point Wightman function we thus have the result $W_{AB}(x, y; *) = W_{BA}(y, x; *)$ for space-like separation between them. Therefore WLC impels

$$W_{AB}(\xi; *) = W_{BA}(-\xi; *) \quad \text{for } \xi^2 < 0. \tag{13}$$

We emphasize again that the symbol $*$ in the above equations and expressions should be inferred on the lines of Eqs. (10) and (12) and should not be confused with the $*$ appearing in the formal expression of the Moyal product. It is just to represent the quantities in non-commutative field theory and distinguish them from their counterparts in the ordinary theory and all the information related to the fact that we are now dealing with a quantum theory on a non-commutative space is coded in the suitably defined spectral density and similar objects of interest.

Also, if ξ is space-like then under a Lorentz transformation $\xi \to -\xi$, $W_{AB}(\xi; *) = W_{AB}(-\xi; *)$. Consider now the following quantity:

$$W_{[AB]}(\xi; *) = W_{AB}(\xi; *) - W_{BA}(\xi; *). \tag{14}$$

Clearly for space-like ξ, $W_{[AB]}(\xi; *)$ vanishes if WLC holds. But the two point Wightman function and the permuted one can both be defined as the boundary values of holomorphic functions in a complex plane. Therefore, the vanishing of $W_{[AB]}(\xi; *)$ for space-like ξ on the real axis implies that $W_{[AB]}(\xi; *) = 0$ everywhere. Therefore, we have $W_{AB}(\xi; *) = W_{BA}(\xi; *)$ which is same as SR in terms of Wightman functions. The same analysis can be extended for fermions where the commutators will be replaced by anti-commutators. For the case of n point Wightman function, the steps remain essentially the same.

Non-commutative quantum field theories are afflicted by the phenomenon of UV/IR mixing and related pathologies [16]. The presence of hard divergences due to this effect may spoil the very basis of Wightman formalism completely, rendering no or very little scope for operations like analytic continuation and smooth convergence of power series etc. In order that the quantum theory makes sense physically, such divergences should not manifest themselves in any form. We assume that some mechanism which bypasses the difficulties arising due to UV/IR mixing is at play and that UV/IR mixing poses no serious threat to the results obtained. For the present case it is not required and therefore we do not bother about the details of such an underlying mechanism.

We briefly comment on the types of non-commutativities and possible difficulties relating to them. For space–space non-commutativity, since the star product does not involve time derivatives, there are no subtleties involved in handling the time ordered products. Therefore, it is easier to see that the theories with only space–space non-commutativity preserve the gross features and axioms of the quantum theory to a greater extent. It becomes far more complicated and involved if space–time non-commutativity is present. A careful handling of time ordering has to be

[2] In presenting this argument we have implicitly assumed that the power series has a smooth convergence, thereby allowing analytic continuation to the desired domain. However, if this requirement is not met, then the arguments would have to be correspondingly modified to allow for convergence, if possible, in an appropriately defined limit.

employed because of the presence of an infinite number of time derivatives which can lead to very complicated and non-compact looking results and expressions. To interpret these results would require further care. Moreover, simplistic and naive treatment should be abandoned in order to obtain sensible results. With time being involved in the non-commutativity, more conceptual issues at the level of foundations of quantum theory may creep in and would require a detailed and more careful treatment. However, for the present case, we tend to ignore all such issues and base our arguments on the hope that such issues do not hamper the basic axioms of the quantum field theory that we have employed, thereby allowing us to reach the desired conclusions. An important thing to remember is the fact that the symmetry is much more restricted in the present situation and depending upon the type of non-commutativity, the theory admits a particular symmetry group and associated structure. Therefore, a careful analysis, keeping track of and correctly taking into account the nature and extent of the allowed operations in such a case like analytic continuation etc., should yield these results.

We therefore see that by looking at the behaviour of Wightman functions it is clear that PCT theorem is valid in quantum field defined on non-commutative space, provided we assume the validity of axioms (Ax1)–(Ax5). Nowhere in the whole analysis has any specific form of non-commutativity assumed, except for the assumption that Θ is independent of space–time variables. As in [10], it remains to be explored in detail whether some specific form of non-commutativity leads to violation of spin-statistic relation in context of Wightman function approach as well. Also, the more conceptual and philosophical issues concerning the time direction being non-commutative have still to be explored and investigated in detail. However, it is very clear that the results obtained hold without much doubt if we restrict ourselves to space-like non-commutativity.

Note added

While this work was completed a similar work [17] appeared. The results more or less match, at least in the domain of validity of the axioms that we have based our arguments on.

Acknowledgements

I would like to thank E. Harikumar and Kamal Datta for their comments.

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